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THE POINT MOMENT AT AN ARBITRARY POINT OF AN ELASTIC PLANE WEAKENED BY AN ELLIPTICAL HOLE†

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Expressions are obtained for the complex potentials characterizing the stress–strain state of an elastic plane with an elliptic hole subjected to a moment at an arbitrary point of the plane. The shear stresses at the edge of the hole are calculated. The stress intensity factors for the limiting case of a straight slit (a crack) are determined. © 1999 Elsevier Science Ltd. All rights reserved.

Expressions have been obtained for the stress intensity factors due to the action of a concentrated moment at an arbitrary point of a plane by considering the effect of a concentrated force on the crack contour [1]. The expression obtained by replacing the complex conjugate coordinate of the point of application of the moment in this result by the value of the mapping function for the crack and multiplying the expressions for the stress intensity factors by $\pi^{-1/2}$ (the stress components were given by other formulae in [1]) is derived in a different way here and is given at the end of the paper.

The complex potentials characterizing the stress-strain state of an infinite plane weakened by an elliptical hole due to the effect of a moment M applied at an arbitrary point Z_0 of the plane outside the hole can be represented in the following form [2]

$$\varphi_1(Z)=\varphi_0^1(Z),\quad \psi_1(Z)=\frac{iM}{2\pi}\frac{1}{Z-Z_0}+\psi_0^1(Z) \tag{1}$$

where the functions $\varphi_0^1(Z)$ and $\psi_0^1(Z)$ are holomorphic outside the hole.

To solve the problem we map the exterior of the ellipse into the interior of the unit circle

$$Z=\omega(\xi)=R\left(\xi+\frac{m}{\xi}\right),\quad R=\frac{a+b}{2},\quad m=\frac{a-b}{a+b} \tag{2}$$

(a and b are the semi-axes of the ellipse). Substituting expression (2) into relations (1), we obtain

$$\varphi(\xi)=\varphi_0(\xi),\quad \psi(\xi)=\frac{iM}{2\pi}\frac{A_0}{\xi-\xi_0}+\psi_0(\xi) \tag{3}$$

$$A_0=\frac{1}{\omega'(\xi_0)},\quad \omega'(\xi_0)=R\left(1-\frac{m}{\xi_0^2}\right),\quad \xi=re^{i\theta},\quad \xi_0=r_0e^{i\theta_0}$$

$$\varphi_0(\xi)=\frac{a_1}{\xi}+\frac{a_2}{\xi^2}+\ldots,\quad \psi_0(\xi)=b_0+\frac{b_1}{\xi}+\frac{b_2}{\xi^2}+\ldots$$

The conditions on the contour have the form

$$\varphi(\sigma)+\frac{\omega(\sigma)}{\overline{\omega}'(1/\sigma)}\overline{\varphi}'\left(\frac{1}{\sigma}\right)+\overline{\psi}\left(\frac{1}{\sigma}\right)=0 \tag{4}$$

$$\overline{\varphi}\left(\frac{1}{\sigma}\right)+\frac{\overline{\omega}(1/\sigma)}{\omega'(\sigma)}\varphi'(\sigma)+\psi(\sigma)=0$$

$$\left(\sigma=e^{i\theta},\quad \frac{\omega(\sigma)}{\overline{\omega}'(1/\sigma)}=\frac{1}{\sigma}\frac{\sigma^2+m}{1-m\sigma^2},\quad \frac{\overline{\omega}(1/\sigma)}{\omega'(\sigma)}=\sigma\frac{1+m\sigma^2}{\sigma^2-m}\right)$$

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Substituting the corresponding values of the complex potentials (3) into (4), and integrating over the unit circle γ we obtain

$$\varphi_0(\xi) = -\frac{1}{2\pi i}\int_\gamma \frac{f_1^0 + if_2^0}{\sigma - \xi}d\sigma \tag{5}$$

$$\psi_0(\xi) = -\frac{1}{2\pi i}\int_\gamma \frac{f_1^0 - if_2^0}{\sigma - \xi}d\sigma - \xi\frac{1+m\xi^2}{\xi^2 - m}\varphi_0'(\xi)$$

$$f_1^0 + if_2^0 = \frac{iM}{2\pi}\frac{\bar{A}_0\sigma}{1-\bar{\xi}_0\sigma}, \quad f_1^0 - if_2^0 = -\frac{iM}{2\pi}\frac{A_0}{\sigma - \xi_0}$$

Evaluating the integrals and substituting the results into (3), we finally obtain

$$\varphi(\xi) = \frac{iM}{2\pi}\frac{\bar{A}_0/\bar{\xi}_0}{1-\bar{\xi}_0\xi}, \quad \psi(\xi) = \frac{iM}{2\pi}\left\{\frac{A_0}{\xi - \xi_0} - \frac{1+m\xi^2}{\xi^2 - m}\frac{\bar{A}_0\xi}{(1-\bar{\xi}_0\xi)^2}\right\} \tag{6}$$

For a circular hole ($m = 0$) we have $A_0 = \bar{A}_0 = 1/R_0$, where R_0 is its radius. The case $m = 1$ corresponds to the complex potentials for a straight slit.

For the shear stress on the contour of an elliptical hole the formula due to Kolosov and Muskhelishvili yields

$$\sigma_\theta^c = \frac{2M}{\pi R^2 \Delta}\{r_0^2[r_0^2 - m\cos 2\theta_0]\{r_0[r_0\sin 2(\theta - \theta_0) - 2\sin(\theta - \theta_0)] + \tag{7}$$

$$+ m[\sin 2\theta - 2r_0\sin(\theta + \theta_0) + r_0^2\sin 2\theta_0]\} - mr_0^2\sin 2\theta_0 \times$$

$$\times\{1 - 2r_0\cos(\theta - \theta_0) + r_0^2\cos 2(\theta - \theta_0) - m[\cos 2\theta - 2r_0\cos(\theta + \theta_0) + r_0^2\cos 2\theta_0]\}\}$$

$$\Delta = \rho^4(r_0^4 - 2mr_0^2\cos 2\theta_0 + m^2)(1 - 2m\cos 2\theta + m^2)$$

$$\rho^2 = 1 - 2r_0\cos(\theta - \theta_0) + r_0^2$$

For a circular hole

$$\sigma_\theta^c = 2M(\pi R_0^2)^{-1}r_0[r_0\sin 2(\theta - \theta_0) - 2\sin(\theta - \theta_0)]/\rho^4 \tag{8}$$

When $m = 1$, expression (7) corresponds to the value σ_θ^c for the contour of a straight slit (a crack). In this case the stress intensity factors are $K_{\rm I}$, $K_{\rm II}$ in the immediate vicinity of the crack tip [3].

It is shown in [4, 5], which deal with the mathematical theory of brittle fracture, that in a large number of cases of practical importance fracture is quasi-brittle, in that while there is a plastic region, it is small in size and concentrated in the immediate vicinity of the crack surfaces. This important idea suggests the possibility of applying the theory of brittle fracture to practical problems.

In this case the stress intensity factors are given in the form

$$K_{\rm I} - iK_{\rm II} = 2\sqrt{2}\lim_{Z\to a}[(Z-a)^{1/2}\varphi'(Z)]$$

In the mapped plane

$$Z = \frac{a}{2}\left(\xi + \frac{1}{\xi}\right)$$

we obtain

$$K_{\rm I} - iK_{\rm II} = \frac{2}{\sqrt{a}}\varphi'(1) = \frac{iM}{\pi\sqrt{a}}\frac{\bar{A}_0}{(1-\bar{\xi}_0)^2}, \quad \bar{A}_0 = \frac{2\bar{\xi}_0}{a(\bar{\xi}_0^2 - 1)} \tag{9}$$

In the case where the moment is applied on the contour in the middle of the upper side of the crack, corresponding to $\theta_0 = \pi/2$, $\xi_0 = e^{i\pi 2} = i$, we will have

$$K_{\rm I} = M\Big/\left(2\pi a^{3/2}\right), \quad K_{\rm II} = 0$$

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